



K21P 1069

No. :

e :

II Semester M.Sc. Degree (CBSS – Reg./Suppl./Imp.) Examination,
October 2021

(2018 Admission Onwards)

MATHEMATICS**MAT3C12 : Functional Analysis**

: 3 Hours

Max. Marks : 80

PART – A.

Answer **four** questions from this **Part**. **Each** question carries **4** marks.

Give an example of an element in l^1 but not in c_{00} and prove your claim.

Explain Riesz lemma for the normed space \mathbb{R}^2 and its subspace $\{(x, x) \in \mathbb{R}^2\}$ geometrically.

Show that the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ on K^n , $n = 1, 2, \dots$ are equivalent.

Show that c_{00} is not a Banach space.

Show that a closed map need not be continuous.

Let $H = l^2$ and for $n = 1, 2, \dots$ and $u_n = (0, \dots, 0, 1, 0, 0, \dots)$ where 1 occurs only in the n^{th} entry. Show that $\{u_n : n = 1, 2, \dots\}$ is an orthonormal basis for H .

PART – B

Answer **4** questions from this **Part** without omitting any **Unit**. **Each** question carries **4** marks.

Unit – I

a) Define L^p - space for $1 \leq p \leq \infty$.

b) Let Y be a closed subspace of a normed space X . For $x + Y$ in the quotient space X/Y let $\|x + Y\| = \inf\{\|x + y\| : y \in Y\}$. Then prove that $\|\cdot\|$ is a norm on X/Y . Also show that a sequence $x_n + Y$ converges to $x + Y$ in X/Y if and only if there is a sequence (y_n) in Y such that $(x_n + y_n)$ converges to x in X .

P.T.O.

8. a) Let $\|\cdot\|$ and $\|\cdot\|'$ be norms on a linear space X . Then prove that the norm $\|\cdot\|$ is stronger than the norm $\|\cdot\|'$ if and only if there is some $\alpha > 0$ such that $\|x\|' \leq \alpha\|x\|$ for all $x \in X$.
- b) Let X and Y be normed spaces and $F : X \rightarrow Y$ be a linear map such that the range $R(F)$ of F is finite dimensional. Then prove that F is continuous if and only if the zero space $Z(F)$ of F is closed in X .
9. a) Let X be a linear space over \mathbb{C} . Regarding X as a linear space over \mathbb{R} , consider a real linear functional $u : X \rightarrow \mathbb{R}$. Define $f(x) = u(x) - iu(ix)$, $x \in X$. Then prove that f is a complex linear functional on X .
- b) State and prove Hahn-Banach extension theorem.

Unit – II

10. a) State and prove Uniform boundedness principle.
- b) State closed graph theorem.
11. a) Show that the closed graph theorem may not hold if the domain of the linear map is not a Banach space.
- b) Let X and Y be normed spaces and $F : X \rightarrow Y$ be linear. Then prove that F is an open map if and only if there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma \|y\|$.
12. a) Prove that there are scalars $k_n \in K$, $n = 0, \pm 1, \pm 2, \dots$ such that $k_n \rightarrow 0$ as $n \rightarrow \pm\infty$, but there is no $x \in L^1([-\pi, \pi])$ such that $\hat{x} = k_n$ for all $n = 0, \pm 1, \pm 2, \dots$.
- b) State two norm theorems.

Unit – III

13. a) State and prove Schwarz inequality.
b) Let $\{u_\alpha\}$ be an orthonormal set in an innerproduct space X and $x \in X$. Let $E_\alpha = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$. Then prove that E_α is a countable set. Also prove that $\langle x, u_\alpha \rangle \rightarrow 0$ as $n \rightarrow \infty$ if E_α is a denumerable set.
14. a) Let E be an orthogonal subset of X and $0 \notin E$. Then prove that E is independent. Also prove that $\|x - y\| = \sqrt{2}$ for all $x \neq y$ in E if E is orthonormal.
b) State and prove projection theorem.
15. a) Define weak boundedness and give an example.
b) Prove that a subset of a Hilbert space is weak bounded if and only it is bounded.
-